

WIENER-CHAOS APPROACH TO OPTIMAL PREDICTION

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ABSTRACT. In this work we combine Wiener chaos expansion approach to study the dynamics of a stochastic system with the classical problem of the prediction of a Gaussian process based on part of its sample path. This is done by considering special bases for the Gaussian space \mathcal{G} generated by the process, which allows us to obtain an orthogonal basis for the Fock space of \mathcal{G} such that each basis element is either measurable or independent with respect to the given samples. This allows us to easily derive the chaos expansion of a random variable conditioned on part of the sample path. We provide a general method for the construction of such basis when the underlying process is Gaussian with stationary increment. We evaluate the bases elements in the case of the fractional Brownian motion, which leads to a prediction formula for this process.

1. INTRODUCTION

Our starting point is a second order Gaussian stationary-increment process $\{X(t), t \in \mathbb{R}\}$, over the probability space $\mathbf{L}_2(\Omega, \mathcal{F}, \mathbb{P})$, with spectral measure Δ . Let \mathcal{G} denote the Gaussian Hilbert space generated by this process, and \mathcal{F}_A the sigma field induced by $\{X(t), t \in A\}$, where A is a Borel set. Given $Y \in \mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P})$, we present a new way to obtain the conditional expectation $\mathbb{E}[Y|\mathcal{F}_A]$ which is based on the Wiener chaos written in terms of a special basis for \mathcal{G} and the Hermite polynomials. Writing

$$(1.1) \quad Y(t) = \sum_{\alpha \in \mathcal{J}} y_{\alpha}(t) H_{\alpha}$$

we have

$$(1.2) \quad \mathbb{E}[Y(t)|\mathcal{F}_A] = \sum_{\alpha \in \mathcal{J}_0} y_{\alpha}(t) H_{\alpha},$$

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where \mathcal{J}_0 depends only on A and Δ . That is, each chaos element H_α is either measurable with respect to \mathcal{F}_A or independent with respect to it.

Surprisingly, this natural decomposition of the space \mathcal{G} , with respect to a variety of prediction problems, has received little treatment in the literature, if any.

If $Y \in \mathcal{G}$, the problem of computing $\mathbb{E}[Y|\mathcal{F}_A]$ reduces to the problem of orthogonal projection onto the closed linear span of the functions $\{X(s), s \in A\}$, denoted by \mathcal{G}_A . When $Y = X(t)$ with $t > 0$ and $A = (-\infty, 0]$, this is the classical Wiener-Kolmogorov prediction problem. If $t > T$ and $A = [-T, T]$, this is the finite horizon prediction problem, which was solved by Krein; see [11]. In this work, we consider these two cases and employ similar methods to obtain an orthonormal basis for the space \mathcal{G}_A . We also note that another case of interest is the interpolation problem, when $t \in (-T, T)$ and $A = (-\infty - T] \cup [T, \infty)$. This problem was solved by Dym and McKean [11]. We also refer to this book for background material on these various problems.

In the more general case where Y is of the form $Y = f(U)$ where $U \in \mathcal{G}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ measurable function such that $f(U) \in \mathbf{L}_2(\Omega, \mathcal{F}(\mathcal{G}), \mathbb{P})$, it follows from see [19] that

$$(1.3) \quad \mathbb{E}[f(U)|\mathcal{F}_A] = \int_{\mathcal{G}} f(\mathbb{E}[U|\mathcal{F}_A](w) + (I - P)(w)) d\mu(w),$$

where $d\mu$ is the standard Gaussian measure on \mathcal{G} and P is the orthogonal projection from \mathcal{G} onto \mathcal{G}_A . If \mathcal{G}_A is one dimensional, then (1.3) reduces to

$$\mathbb{E}[f(U)|\mathcal{F}_A] = f_M(\mathbb{E}[U|\mathcal{F}_A]),$$

where $f_M(U)$ is the Mehler transform of f ; see [20, Ex. 4.18]. In general, formula (1.3) does not lead to easy computations because of the Gaussian integral.

A more simple case of non-linearity is $Y = U^n$ for $U \in \mathcal{G}$ and natural n , in which $\mathbb{E}[Y|\mathcal{F}_A]$ is the n_{th} moment of the the random variable U with respect to the conditional Gaussian distribution

$$f_{U|\mathcal{F}_A}(u) = \frac{1}{\sqrt{2\pi\sigma_{MSE}^2}} \exp \left\{ -\frac{(u - \mathbb{E}[U|\mathcal{F}_A])^2}{2\sigma_{MSE}^2} \right\},$$

where $\sigma_{MSE}^2 = \mathbb{E}[(U - \mathbb{E}[U|\mathcal{F}_A])^2]$. Thus

$$(1.4) \quad \mathbb{E}[U^n|\mathcal{F}_A] = h_n^{[-\sigma_{MSE}^2]}(\mathbb{E}[U|\mathcal{F}_A]),$$

where $h_n^{[\alpha]}(x)$ is the n_{th} Hermite polynomial with parameter α :

$$(1.5) \quad h_n^{[\alpha]}(x) \triangleq n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{\alpha^m x^{n-2m} \cdot (-1/2)^m}{m!(n-2m)!}.$$

Relation (1.4) can be reformulate as

$$(1.6) \quad \mathbb{E}[h_n(U)|\mathcal{F}_A] = h_n^{[\sigma_{MSE}^2]}(E[U|\mathcal{F}_A]),$$

where we denote $h_n = h_n^{[1]}$. In a way, formulas (1.4) and (1.6) may be seen as transforming the non-linear problem of finding $\mathbb{E}[Y|\mathcal{F}_A]$ into the linear problem of finding $\mathbb{E}[U|\mathcal{F}_A]$, since the latter is an element of the linear space \mathcal{G} . This approach can be generalized for any $Y \in \mathbf{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ if we decompose it as a sum of polynomials in elements of \mathcal{G} . This is the celebrated Wiener chaos decomposition.

By introducing a special family of chaos elements H_α , we can write the chaos expansion of $f(U)$ and obtain from (1.2) the chaos expansion of $\mathbb{E}[f(U)|\mathcal{F}_A]$. The resulting expansion allows us to recover the statistics of $\mathbb{E}[f(U)|\mathcal{F}_A]$, and moreover to obtain a representation of its elements directly from the path $\{X(t), t \in A\}$. Thus we replace the non-linear problem of finding (1.3) by the linear problem of expressing an element of \mathcal{G} by the samples $\{X(t), t \in A\}$. This will be discussed in greater details in Section 6.

If $\{E_k, k \in J\}$, where $J \subset \mathbb{Z}$, is an orthogonal basis for the Gaussian Hilbert space \mathcal{G} , then a standard recipe for obtaining a basis for $\mathbf{L}_2(\Omega, \mathcal{F}(\mathcal{G}), \mathbb{P})$ is as follows [20, 16, 15]: let \mathcal{J} be the set of multi-indexes over J , i.e. the set of functions $J \rightarrow \mathbb{N}$ with compact support. For $\alpha = (\dots \alpha_{j_1}, \alpha_{j_2}, \dots) \in \mathcal{J}$, define

$$H_\alpha(\omega) = \prod_{j \in J} h_{\alpha_j}(E_j(\omega)),$$

where $\{h_n, n \geq 0\}$ are the Hermite polynomials

$$(1.7) \quad h_n(x) \triangleq n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2m} \cdot (-1/2)^m}{m!(n-2m)!}.$$

Assume moreover that $\{E_j, j \in J_0\}$, $J_0 \subset J$ is an orthogonal basis for \mathcal{G}_A and denote by \mathcal{J}_0 the subset of multi-indexes whose support is contained in J_0 . Our underlying observation is given by the following theorem.

Theorem 1.1. *For every $\alpha \in \mathcal{J}$, H_α is measurable with respect to \mathcal{F}_A if and only if $\alpha \in \mathcal{J}_0$, i.e. the support of α is contained in $J_0 \subset \mathbb{Z}$.*

Theorem 1.1 can be implicitly found in [20, Ch. 7], and an explicit proof will be given in Section 2. Theorem 1.1 might have been useless unless we could obtain some explicit orthogonal bases for the space \mathcal{G}_A , this is the content of Sections 4 and 5 in which we review some methods to do so in two cases of interest for the time index set A . The setting for Sections 4 and 5 is given in Section 2. In Section 6 we discuss on some application of these chaos elements, together with some examples for the case where X is the fractional Brownian motion.

2. PROOF OF THEOREM 1.1

Theorem 2.1. *Let $\{E_k, k \in J\}$ be an orthonormal basis for \mathcal{G} such that $\{E_j, j \in J_0\}$ span \mathcal{G}_A where $J_0 \subset J$. Then for every $\alpha \in \mathcal{J}$, H_α is measurable with respect to \mathcal{F}_A if and only if α is contained in $\mathcal{J}_0 \subset \mathbb{Z}$.*

Proof. Let $\Gamma(\mathcal{G})$ be the symmetric Fock space of \mathcal{G} . Recall that we have [20, p. 18]

$$\Gamma(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \mathcal{G}^{:n:} = \mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P}),$$

where $\mathcal{G}^{:n:}$ is the n_{th} symmetric tensor power of \mathcal{G} . We also denote by

$$: X_1, \dots, X_m :$$

the Wick product of the elements X_1, \dots, X_m of \mathcal{G} .

Let P denote the orthogonal projection onto \mathcal{G}_A , i.e. for an element $X \in \mathcal{G}$ we have

$$PX = \mathbb{E}[X | \mathcal{F}_A].$$

Since $\|P\| = 1$, ΓP , the second quantization of P , is a bounded linear operator on $\Gamma(\mathcal{G})$ [20, Theorem 4.5], and by [20, Theorem 4.9] we have that $\Gamma P : \mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P}) \rightarrow \mathbf{L}_2(\Omega, \mathcal{F}_A, \mathbb{P})$ equals the conditional expectation

$$Y \rightarrow \mathbb{E}[Y | \mathcal{F}_A], \quad Y \in \mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P}).$$

We have

$$\begin{aligned}
\mathbb{E}[H_\alpha|\mathcal{F}_A] &= (\Gamma P)H_\alpha \\
&= \Gamma P \prod_{j \in J} h_{\alpha_j}(E_j) \\
&\stackrel{a}{=} \Gamma P : \prod_{j \in J} E_j^{\alpha_j} : \\
(2.8) \quad &\stackrel{b}{=} : \prod_{j \in J} P E_j^{\alpha_j} : \\
&= \prod_{j \in J} h_{\alpha_j}(P E_j) \\
&= \prod_{j \in J_0} h_{\alpha_j}(P E_j) \prod_{j \in J \setminus J_0} h_{\alpha_j}(P E_j).
\end{aligned}$$

where (a) follows from [20, Theorem 3.21] and (b) follows from the definition of the second quantization of P (see [20, Theorem 4.5]). Since $E_j \in \mathcal{G}_A^\perp$ for $j \in J \setminus J_0$, if $\alpha_j \neq 0$ for some entry of α then $h_{\alpha_j}(P E_j) = h_{\alpha_j}(0) = 1$. In this case $\mathbb{E}[H_\alpha|\mathcal{F}_A] = 0$, which means that H_α is independent of \mathcal{F}_A . The other option is that $\alpha_j = 0$ for all $j \in J \setminus J_0$. Since for $h_{\alpha_j}(x) \equiv 1$ by definition if $\alpha_j = 0$, this case leads to

$$\mathbb{E}[H_\alpha|\mathcal{F}_A] = \prod_{j \in J_0} h_{\alpha_j}(P E_j) = \prod_{j \in J_0} h_{\alpha_j}(E_j) = H_\alpha.$$

□

3. HILBERT SPACES ASSOCIATED WITH A GAUSSIAN STATIONARY INCREMENT PROCESS

In this section we review standard ideas from the literature on continuous time Gaussian stochastic processes. We describe two additional Hilbert spaces isomorphic to \mathcal{G} , using the notions of the *Wiener integral* and the *trigonometric isomorphism*. This sets the frameworks for sections 4 and 5 in which we obtain a basis for \mathcal{G} that satisfy the conditions in Theorem 2.1.

Assume first we are given a Gaussian stationary process $\dot{X}(\cdot) \triangleq \{\dot{X}(t), t \in \mathbb{R}\}$. The spectral measure $\Delta(\gamma)$ is determined by Bochner's theorem through

$$(3.9) \quad \mathbb{E}[\dot{X}(t_1)\dot{X}(t_2)] = \int_{-\infty}^{\infty} e^{i\gamma(t_1-t_2)} d\Delta(\gamma).$$

This defines the so called *trigonometric isomorphism* between the Gaussian Hilbert space \mathcal{G} generated by $\dot{X}(\cdot)$, i.e. the close linear span of

$\{\dot{X}(t), t \in \mathbb{R}\}$ in $\mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P})$, and the space $\mathbf{L}_2(d\Delta)$, given by

$$\dot{X}(t) \longrightarrow e^{i\gamma t}.$$

If $\dot{X}(\cdot)$ is path-wise integrable then $X(t) \triangleq \int_0^t \dot{X}(s)ds$ is a Gaussian stationary increment process, with covariance function

$$(3.10) \quad \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \frac{1 - e^{i\gamma t_1}}{i\gamma} \frac{1 - e^{-i\gamma t_2}}{-i\gamma} d\Delta(\gamma),$$

so that $\frac{e^{i\gamma t}-1}{i\gamma}$ is the image of $X(t)$ under the trigonometric isomorphism. In the setting of distributions, we can write the following correspondences

$$(3.11) \quad \begin{aligned} \dot{X}(t) &\longleftrightarrow e^{i\gamma t} &\longleftrightarrow \delta(t - \cdot), \\ X(t) &\longleftrightarrow \frac{e^{i\gamma t}-1}{i\gamma} &\longleftrightarrow \mathbf{1}_t, \end{aligned}$$

where the left relation is the trigonometric isomorphism and the right relation is the Fourier transform. In (3.11) we used $\delta(t)$ to denote the Dirac delta distribution concentrated at the origin and

$$\mathbf{1}_t(x) \triangleq \mathbf{1}_{[0,t]}(x) \triangleq \begin{cases} 1, & x \in [0, t] \\ 0, & x \notin [0, t], \end{cases}$$

We see that for a given $t \geq 0$, $X(t)$ may be interpreted as the stochastic integral of the deterministic function $\mathbf{1}_t$ [20, p. 87], and can be extended to $t < 0$ by setting

$$\mathbf{1}_t(x) = \begin{cases} 1 & 0 < x \leq t, \\ -1 & -t \leq x < 0, \\ 0 & \text{otherwise} \end{cases}.$$

For $f \in \mathbf{L}_2(\mathbb{R})$ we denote by \hat{f} its Fourier transform

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(t) e^{i\gamma t} dt$$

and by \check{f} its inverse.

In general, for $f \in \mathbf{L}_2(\mathbb{R})$ subject to

$$(3.12) \quad \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\Delta(\gamma) < \infty,$$

we can define its stochastic integral with respect to the process $X(\cdot)$ as the zero mean Gaussian random variable with variance $\int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\Delta(\gamma)$.

The set of functions in $\mathbf{L}_2(\mathbb{R})$ which satisfy (3.12) constitute a pre-Hilbert space, and we denote its completion by \mathbf{L}_Δ . The map $I : \mathbf{L}_\Delta \longrightarrow \mathcal{G}$ that carries an element of \mathbf{L}_Δ into its stochastic integral is an Hilbert space isomorphism, and \mathcal{G} can be regarded as the Gaussian Hilbert spaces associated with the Hilbert space \mathbf{L}_Δ [20], in the sense that for each $f_1, \dots, f_n \in \mathbf{L}_\Delta$, $I(f_1), \dots, I(f_n)$ have a joint central normal distribution with covariance matrix Q ,

$$Q_{j,i} = (f_i, f_j)_\Delta = \int_{-\infty}^{\infty} \widehat{f}_i(\gamma) \overline{\widehat{f}_j(\gamma)} d\Delta(\gamma),$$

where $(\cdot, \cdot)_\Delta$ is the inner product in \mathbf{L}_Δ induced by the norm (3.12). Using these notations, the covariance function (3.10) can be written as

$$\mathbb{E}[X(t)X(s)] = (\mathbf{1}_t, \mathbf{1}_s)_\Delta.$$

In the case of $d\Delta = d\gamma$, \mathbf{L}_Δ reduces to $\mathbf{L}_2(\mathbb{R})$ and the image of $f \in L_2(\mathbb{R})$ under I is called the Wiener stochastic integral of f [7, Chapter 9].

Remark 3.1. *We note that for $f \in \mathbf{L}_\Delta$, one can define its stochastic integral with respect to X in the usual way by first setting*

$$\int_{\mathbb{R}} f(t) dX(t) = \sum_i \alpha_i (X(t_{i+1}) - X(t_i))$$

for a simple function $f(t) = \sum_i \alpha_i \mathbf{1}_{[t_{i+1}, t_i]}$, and then take the limit in \mathbf{L}_Δ for a general $f \in \mathbf{L}_\Delta$. It can be shown that we obtain

$$(3.13) \quad \int_{\mathbb{R}} f(t) dX(t) = I(f)$$

in $\mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P})$, that is, both definitions coincide.

In many practical cases, almost every sample path of the stationary increment process $X(\cdot)$ is nowhere differentiable. This happens for example in the case of the Brownian motion or the fractional Brownian motion. However under the condition

$$(3.14) \quad \int_{\mathbb{R}} \frac{d\Delta(\gamma)}{1 + \gamma^2} < \infty,$$

it is easy to see that the indicator function $\mathbf{1}_t$ still belongs to \mathbf{L}_Δ . Since both spaces \mathcal{G} and \mathbf{L}_Δ , as well as the isometric map between them, are determined exclusively by the spectral measure Δ , starting with Δ , we may use the representation

$$X(t) = I(\mathbf{1}_t), \quad t \in \mathbb{R}$$

as the definition of the process X . We set

$$z_t = T(\mathbf{1}_t) = \frac{e^{i\gamma t-1}}{i\gamma}, \quad t \in \mathbb{R}.$$

Under the condition (3.14), each z_t belongs to $\mathbf{L}_2(d\Delta)$. Denote by \mathbf{Z} the close linear span of $\{z_t, t \in \mathbb{R}\}$ in $\mathbf{L}_2(d\Delta)$ and by \mathbf{Z}_A the close linear span of $\{z_t, t \in A\}$ in $L_2(d\Delta)$. It is well known (see for example [11]) that $\mathbf{Z} = \mathbf{L}_2(d\Delta)$.

We have obtained the following isomorphic Hilbert spaces

$$\mathbf{L}_\Delta \xrightarrow{I} \mathcal{G} \xrightarrow{T} \mathbf{Z}.$$

Note that in the sense of distributions, $T \circ I : \mathbf{L}_\Delta \rightarrow \mathbf{Z}$ is the Fourier transform.

The importance of the above Hilbert spaces isomorphism is that it allows us to exchange the problem of obtaining an orthogonal basis for \mathcal{G} and \mathcal{G}_A with the problem of doing so in \mathbf{Z} and \mathbf{Z}_A . Our benefit comes from the fact that now the theory of orthogonal projections into spaces of analytical functions is at our disposal.

4. PREDICTION WITH RESPECT TO THE ENTIRE PAST

In order to be in the setting of Theorem 1.1, we first need to find an explicit orthogonal basis for the space \mathcal{G}_A . In this section we will show how to do so in the case that $A = (-\infty, 0]$ which corresponds to the Wiener-Kolmogorov prediction problem. The case where $A = [-T, T]$ for some $T > 0$ which corresponds to the problem solved by Krein is treated in the next section.

Recall that in the case of prediction with respect to the entire past, Szegő theorem provides us with a criterion whether the prediction is perfect or not, i.e. when

$$\mathbb{E}[X(t)|\mathcal{F}_{(-\infty, 0]}] = X(t), \quad \forall t \in \mathbb{R},$$

is in $\mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P})$ or not. Or in trigonometric language: whether $z_t \in \mathbf{Z}_{(-\infty, 0]}$ or else

$$\mathbb{E}\left[\left(X(t) - \mathbb{E}[X(t)|\mathcal{F}_{(-\infty, 0]}]\right)^2\right] > 0.$$

Szegő criterion says that if

$$(4.15) \quad \int_{-\infty}^{\infty} \frac{\log \Delta'(\gamma)}{\gamma^2 + 1} d\gamma > -\infty,$$

then $\mathbf{Z}_{(-\infty,0]} \neq \mathbf{Z}$, and in particular $z_T \notin \mathbf{Z}_{(-\infty,0]}$ for any $T > 0$. The other option

$$(4.16) \quad \int_{-\infty}^{\infty} \frac{\log \Delta'(\gamma)}{\gamma^2 + 1} d\gamma = -\infty,$$

implies $\mathbf{Z}_{(-\infty,0]} = \mathbf{Z}$, i.e., the future is completely determined by the past.

Karhunen(1950) has showed that under the conditions (4.15) and $\Delta(\infty) = \int_{-\infty}^{\infty} d\Delta(\gamma) < \infty$, the spectral density can be decomposed as

$$(4.17) \quad \Delta'(\gamma) = h(\gamma)\bar{h}(\gamma),$$

where h is an *outer function* in the Hardy space H^{2+} (see [2]). An outer function $h \in H^{2+}$ satisfies the property that the span of $e^{i\gamma t}\bar{h}(\gamma)$, $t < 0$, in $\mathbf{L}_2(d\gamma)$ equals H^{2-} , or equivalently, that the span of $z_t\bar{h}$, $t < 0$, in $\mathbf{L}_2(\mathbb{R})$ equals H^{2-} . To see this equivalence we note that, for $k \in \mathbf{L}_1(\mathbb{R})$

$$\int_{\mathbb{R}} e^{i\gamma t} k(\gamma) d\gamma = 0, \forall t > 0 \iff \int_{\mathbb{R}} \frac{e^{i\gamma t} - 1}{\gamma} k(\gamma) d\gamma = 0, \forall t > 0$$

as is seen by differentiation and integration with respect to t .

Throughout this section we assume the spectral measure Δ satisfies both condition (3.14) and Szegő criterion for optimal prediction with respect to the entire past. We further assume that $d\Delta$ is absolutely continuous with respect to the Lebesgue measure, namely $d\Delta = \Delta'(\gamma)d\gamma$. In view of the discussion in [11, Section 4.3], this assumption does not limit the generality of our approach. Since these assumption does not yet guaranty $\Delta(\infty) < \infty$, we look instead at the measure $\frac{\Delta'(\gamma)d\gamma}{1+\gamma^2}$. We have

$$\int_{-\infty}^{\infty} \frac{\log \left(\frac{\Delta'(\gamma)}{1+\gamma^2} \right)}{1+\gamma^2} d\gamma < \infty,$$

so that we may decompose $\frac{\Delta'(\gamma)d\gamma}{1+\gamma^2}$ as

$$(4.18) \quad \frac{\Delta'(\gamma)d\gamma}{1+\gamma^2} = h(\gamma)\bar{h}(\gamma)$$

with h outer, and thus obtain the decomposition:

$$d\Delta = \Delta'(\gamma)d\gamma = |(\gamma - i)h(\gamma)|^2 d\gamma.$$

Since the process X is real, Δ is always even and we can also impose the condition $h(-\gamma) = -\bar{h}(\gamma)$ [11, Exercise 2.7.4], so that the inverse Fourier transform of $(\gamma - i)h$ is a real distribution.

Lemma 4.1. *The closed linear span of the functions $\{(\gamma + i)z_t h, t \leq 0\}$ in $\mathbf{L}_2(d\gamma)$ equals H^{2-} .*

Proof. Denote by \mathbf{K} the close linear span of $\{z_t(\gamma + i)h, t \leq 0\}$ in $\mathbf{L}_2(d\gamma)$. Let \mathbf{L} be the set of functions $f \in H^{2-}$ such that $(\gamma - i)f(\gamma)$ still belongs to H^{2-} . It has been noted above that for $h \in H^{2-}$ outer, the closed linear span of $\{z_t h, t \leq 0\}$ in $\mathbf{L}_2(d\gamma)$ is all H^{2-} , so if a function $f \in \mathbf{L}$ satisfies

$$\int_{-\infty}^{\infty} z_t(\gamma)(\gamma + i)h(\gamma)\overline{f}(\gamma)d\gamma = 0,$$

for all $t \leq 0$, we conclude that $(\gamma + i)f \equiv 0$, so $f = 0$, and we have that $\mathbf{K}^\perp \cap \mathbf{L} = \{0\}$. In order to complete the proof it is enough to show that \mathbf{L} is a dense subset of H^{2-} . The Schwartz space \mathcal{S} of smooth rapidly decreasing functions is a dense subset of $\mathbf{L}_2(\mathbb{R})$ and is invariant under differentiation and Fourier transformation. For $s \in \mathcal{S}$ we denote by \tilde{s} its projection into $\mathbf{L}_2((-\infty, 0])$, which is the set of functions in $\mathbf{L}_2(\mathbb{R})$ supported in $(-\infty, 0]$. Recall that $H^{2-} = \widehat{\mathbf{L}_2((-\infty, 0])} = \{\widehat{f}, f \in \mathbf{L}_2((-\infty, 0])\}$, which implies that $\widehat{\mathcal{S}}$ is a dense subset of H^{2-} .

We prove that it is also contained in \mathbf{L} . Let $s \in \mathcal{S}$, then

$$(\gamma + i)\widehat{\tilde{s}} = (\gamma + i) \int_{-\infty}^0 s(t)e^{i\gamma t} dt = i \int_{-\infty}^0 \frac{ds}{dt}(t)e^{i\gamma t} dt + i \int_{-\infty}^0 s(t)e^{i\gamma t} dt,$$

and the last two terms are the Fourier transform of functions in $\mathbf{L}_2((-\infty, 0])$, hence belong to H^{2-} . This completes the proof. \square

In what follows we construct an orthonormal basis for the space \mathbf{L}_Δ in terms of the function h and the functions

$$e_n(\gamma) = \frac{1}{\sqrt{\pi}} \frac{1}{1 - i\gamma} \left(\frac{1 + i\gamma}{1 - i\gamma} \right)^n, n \in \mathbb{Z},$$

which constitute an orthonormal basis of $\mathbf{L}_2(\mathbb{R})$. In addition, the family $\{e_n, n \geq 0\}$ spans the Hardy space H^{2+} while the family $\{e_n, n < 0\}$ spans H^{2-} ; see [11, Section 2.5]. We also note that for $n \geq 0$ and $x > 0$, the inverse Fourier transform of the $\{e_n\}$ are the Laguerre functions:

$$e_n^\vee(x) = \frac{1}{\sqrt{\pi n!}} \frac{d^n}{d\gamma^n} (e^{-i\gamma x} (i - \gamma)^n)$$

evaluated at $\gamma = -i$.

We are now looking for a set of functions in \mathbf{L}_Δ whose images under I constitute an orthogonal basis in \mathcal{G} . In view of (4.18), the function $(\gamma + i)\overline{h}s$, when s belongs to the Schwartz space \mathcal{S} of smooth rapidly decreasing functions, belongs to $\mathbf{L}_2(\mathbb{R})$. Moreover the linear span of

the functions $(\gamma + i)\bar{h}s$ with $s \in \mathcal{S}$ is dense in $\mathbf{L}_2(d\gamma)$. Indeed, let $g \in \mathbf{L}_2(d\gamma)$ be such that

$$\int_{\mathbb{R}} s(\gamma)(\gamma + i)\bar{h}(\gamma)\bar{g}(\gamma)d\gamma = 0, \quad \forall s \in \mathcal{S}.$$

Then the function $(\gamma - i)h\bar{g}$ (which need not belong to $\mathbf{L}_2(d\gamma)$) defines the zero distribution on \mathcal{S} , and so is a.e. equal to 0.

This proves that there exists a sequence $\{s_k\} := \{s_k, k \in \mathbb{N}\}$ of Schwartz functions such that

$$\lim_{k \rightarrow \infty} \|(\gamma + i)\bar{h}s_k - e_n\|_{\mathbf{L}_2(d\gamma)} = 0.$$

Therefore the sequence $\{s_k\}$ tends to $\frac{e_n}{(\gamma+i)h}$ in $\mathbf{L}_2(d\Delta)$, and so the sequence $\{\tilde{s}_k\}$ is a Cauchy sequence in \mathbf{L}_Δ . We denote its limit by ξ_n .

If $\frac{e_n}{(\gamma+i)h}$ is in $\mathbf{L}_2(d\gamma)$, then $\hat{\xi}_n = \frac{e_n}{(\gamma+i)h}$.

Theorem 4.2. *The set $\{I(\xi_n), n \in \mathbb{Z}\}$ forms an orthonormal basis for \mathcal{G} . Moreover,*

$$\mathbb{E}[I(\xi_n)|\mathcal{F}_{(-\infty, 0]}] = \begin{cases} I(\xi_n), & n < 0 \\ 0, & n \geq 0 \end{cases},$$

so that $\{I(\xi_n), n < 0\}$ spans the past, and $\{I(\xi_n), n \geq 0\}$ spans its orthogonal complement.

Proof. The fact that the $I(\xi_n)$ are orthonormal is immediate by construction since

$$\mathbb{E}[I(\xi_n)I(\xi_m)] = \int_{-\infty}^{\infty} \frac{e_n(\gamma)}{(\gamma - i)\bar{h}(\gamma)} \frac{\overline{e_m(\gamma)}}{(\gamma + i)h(\gamma)} (1 + \gamma^2) |h(\gamma)|^2 d\gamma = (e_n, e_m)_{\mathbf{L}_2(d\gamma)}.$$

To show that they span \mathcal{G} , let $f \in \mathbf{L}_\Delta$ and assume that $I(f)$ is perpendicular to their span. Then for all $n \in \mathbb{Z}$,

$$0 = \mathbb{E}[I(\xi_n)I(f)] = \int_{-\infty}^{\infty} e_n(\gamma - i)\bar{f}h d\gamma = \left(e_n, (\gamma + i)\bar{h}\hat{f}\right)_{\mathbf{L}_2(d\gamma)}.$$

Since $\{e_n, n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{L}_2(d\gamma)$, it follows that $\hat{f}\bar{h}$ is zero in $\mathbf{L}_2(d\gamma)$. But it follows from condition (4.15) that $h(\gamma) \neq 0$ almost everywhere, so we conclude that f , and thus $I(f)$, equals zero. Note that this also proves that $\{\xi_n, n \in \mathbb{Z}\}$ is an orthonormal basis of \mathbf{L}_Δ .

Now for $t \leq 0$,

$$\begin{aligned}
 \mathbb{E}[I(\xi_n)X(t)] &= \int_{-\infty}^{\infty} \frac{e_n}{(\gamma+i)\bar{h}}(\gamma^2+1)\bar{z}_t|h(\gamma)|^2 d\gamma \\
 (4.19) \qquad &= \int_{-\infty}^{\infty} (\gamma-i)e_n\bar{z}_t h d\gamma \\
 &= (e_n, (\gamma+i)z_t\bar{h})_{\mathbf{L}_2(d\gamma)}.
 \end{aligned}$$

From Lemma 4.1 we know that the span of $\{z_t(\gamma+i)\bar{h}, t \leq 0\}$ in $\mathbf{L}_2(\mathbb{R})$ equals H^{2-} , so the last term in (4.19) vanishes for $n \geq 0$. In order to calculate $\widetilde{I(\xi_n)} \triangleq \mathbb{E}[I(\xi_n)|\mathcal{F}_{(-\infty,0]}]$ for $n < 0$ we can use the trigonometric isomorphism and instead look for the projection of $\frac{e_n(\gamma)}{(\gamma+i)\bar{h}}$ onto $\mathbf{Z}_{(-\infty,0]}$. Let $g = c_1 z_t + \dots + c_n z_t$, where $t_1, \dots, t_n, \leq 0$ and $c_1, \dots, c_n \in \mathbb{C}$.

$$\begin{aligned}
 \mathbb{E}[|I(\xi_n) - \widetilde{I(\xi_n)}|^2] &= \inf_g \int_{-\infty}^{\infty} \left| \frac{e_n(\gamma)}{(\gamma+i)\bar{h}(\gamma)} - g(\gamma) \right|^2 \Delta'(\gamma) d\gamma = \\
 &\quad \inf_g \int_{-\infty}^{\infty} |e_n(\gamma) - g(\gamma)(\gamma+i)\bar{h}(\gamma)|^2 d\gamma.
 \end{aligned}$$

Since the span of $\{z_t(\gamma+i)\bar{h}, t \leq 0\}$ is H^{2-} and $e_n \in H^{2-}$ for $n < 0$, the last projection norm is trivial, so $\widetilde{I(\xi_n)} = I(\xi_n)$ in \mathcal{G} , and hence $I(\xi_n) \in \mathcal{F}_{(-\infty,0]}$. \square

Example 4.3. *The chaos expansion of $X(t)$ is given by*

$$X(t) = I(\mathbf{1}_t) = \sum_{n=-\infty}^{\infty} (\xi_n, \mathbf{1}_t)_{\Delta} I(\xi_n) = \sum_{n=-\infty}^{\infty} (\xi_n, \mathbf{1}_t)_{\Delta} H_{\epsilon(n)},$$

where $\epsilon(n) = (\dots, 0, 1, 0, \dots)$ with 1 at the n_{th} place. It follows that

$$(4.20) \qquad \mathbb{E}[X(t)|\mathcal{F}_{(-\infty,0]}] = \sum_{n=-\infty}^{-1} (\xi_n, \mathbf{1}_t)_{\Delta} I(\xi_n).$$

The complementary projection is given by

$$\sum_{n=0}^{\infty} (\xi_n, \mathbf{1}_t)_{\Delta} I(\xi_n),$$

so that the variance of the prediction error is

$$\sum_{n=0}^{\infty} (\xi_n, \mathbf{1}_t)_{\Delta}^2.$$

Example 4.4. *The Wick exponent of the process $X(t)$ has the chaos expansion*

$$: e^{X(t)} := \exp \left\{ I(\mathbf{1}_t) - \|\mathbf{1}_t\|_\Delta^2 \right\} = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha,$$

with (see [17, Exercise 2.8 (e)])

$$c_\alpha = \prod_{n=-\infty}^{\infty} \frac{(1_t, \xi_n)_\Delta^{\alpha_n}}{\alpha_n!}.$$

It follows that

$$\mathbb{E} \left[: e^{X(t)} : | \mathcal{F}_{(-\infty, 0]} \right] = \sum_{\alpha \in \mathcal{J}_0} c_\alpha H_\alpha,$$

where now for $\alpha \in \mathcal{J}_0$,

$$c_\alpha = \prod_{n=-\infty}^{-1} \frac{(1_t, \xi_n)_\Delta^{\alpha_n}}{\alpha_n!}.$$

From [20] we have

$$\mathbb{E} \left[: e^U : | \mathcal{F}' \right] =: e^{\mathbb{E}[U | \mathcal{F}']} := \exp \left\{ \mathbb{E}[U | \mathcal{F}'] - \frac{1}{2} \mathbb{E}[X^2 | \mathcal{F}'] \right\}$$

and we obtain the following identity:

$$(4.21) \quad \sum_{\alpha \in \mathcal{J}_0} c_\alpha H_\alpha = \exp \left\{ \sum_{n=-\infty}^{-1} (\xi_n, \mathbf{1}_t)_\Delta I(\xi_n) - \frac{1}{2} \sum_{n=0}^{\infty} (\xi_n, \mathbf{1}_t)_\Delta^2 \right\}.$$

4.1. Basis Elements and Sample Path Relation. In the classical prediction problem we are asked to find the conditional expectation with respect to $\mathcal{F}_{(-\infty, 0]}$ expressed in term of the path $X_{(-\infty, 0]}(\cdot) = \{X(t), t \in A\}$. By Theorem 4.2 for $n < 0$, $I(\xi_n)$ is completely determined by $X_{(-\infty, 0]}(\cdot)$. Due to the trigonometric isomorphism we have

$$I(\xi_n) = I(\mathbf{1}_{[-\infty, 0]}\xi_n) + I(\mathbf{1}_{[0, \infty]}\xi_n),$$

and

$$(4.22) \quad \|\mathbf{1}_{[0, \infty]}\xi_n\|_\Delta^2 = \int_{-\infty}^{\infty} \left| \widehat{\mathbf{1}_{[0, \infty]}\xi_n}(\gamma) \right|^2 d\Delta(\gamma).$$

Note that $\widehat{\mathbf{1}_{[0, \infty]}\xi_n}$ is the projection of

$$(4.23) \quad \widehat{\xi_n}(\gamma) = \frac{e_n(\gamma)}{(\gamma + i)\bar{h}(\gamma)} = \frac{1}{\sqrt{\pi}} \left(\frac{1 + i\gamma}{1 - i\gamma} \right)^n \frac{1}{(1 + \gamma^2)\bar{h}(\gamma)}$$

into H^{2+} (see [11, Ch. 2.4]). If \bar{h} does not vanish too fast as $\gamma \rightarrow \infty$, (in general, condition (4.15) does not guarantee that), then $\widehat{\xi_n}$ belongs

to H^{2-} , in which case $\widehat{1_{[0,\infty]}\xi_n} = 0$. That is, we have the following proposition:

Proposition 4.5. *If $\widehat{\xi_n} \in \mathbf{L}_2(d\gamma)$, then*

$$\mathbb{E} [I(\xi_n) | \mathcal{F}_{(-\infty, 0]}] = I(\xi_n \mathbf{1}_{(-\infty, 0]}),$$

for all $n = -1, -2, \dots$

If the condition in Proposition 4.5 is met, then the $\mathbf{L}_2(\Omega, \mathcal{F}_{\mathbb{R}}, \mathbb{P})$ stochastic integral with respect to the process $X(t)$ can also be evaluated from its sample path which is defined in the standard way as in explained in (3.13). See also [14, Sec. 2] for a pathwise definition of the stochastic integral with respect to the fractional Brownian motion. In such case we get

$$(4.24) \quad I(\xi_n) = I(\mathbf{1}_{(-\infty, 0]}\xi_n) = \int_{-\infty}^0 \xi_n(t) dX(t).$$

This allows us to express the $I(\xi_n)$ in (4.20) in terms of the sample path, which leads to a prediction formula for a general Gaussian stationary increment process.

5. BOUNDED TIME INTERVAL

In the case where $A = [-T, T]$ we are looking for an orthogonal basis for $\mathcal{G}_{[-T, T]}$, or equivalently, for its image under the trigonometric isomorphism $\mathbf{Z}_{[-T, T]}$. For the conditions to optimal prediction in this case we refer to [11, Section 6.4]. The space $\mathbf{Z}_{[-T, T]}$ is a reproducing kernel Hilbert space of entire functions isometrically included in $\mathbf{L}_2(d\Delta)$, and invariant under the backward shift operators

$$R_a f(\gamma) = \frac{f(\gamma) - f(a)}{\gamma - a}, \quad a \in \mathbb{C}.$$

Therefore, by a theorem of de Branges, see [4, Theorem 3], and (for instance) by an application of [1, Theorem 3.1], one sees that the reproducing kernel of $\mathbf{Z}_{[-T, T]}$ is of the form

$$(5.25) \quad K(\gamma, \lambda) = \frac{B(\gamma)\overline{A}(\lambda) - A(\gamma)\overline{B}(\lambda)}{\gamma - \overline{\lambda}},$$

where $A(\gamma)$ and $B(\gamma)$ are entire function of the variable λ of finite exponential type. A characterization of certain orthogonal sets in such spaces is given in [5, Theorem 22]. We recall the result for completeness. Set $E(\gamma) = A(\gamma) - iB(\gamma)$. Then there exists a continuous function $\varphi(x)$

($x \in \mathbb{R}$) such that $E(x, T)e^{i\varphi(x)} \in \mathbb{R}$ for all $x \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ and let $x_1, x_2, \dots \in \mathbb{R}$ be such that $\varphi(x_n) \equiv \alpha \pmod{\pi}$. The functions

$$K(\gamma, x_n)$$

form an orthogonal set of $\mathbf{Z}_{[-T, T]}$, and it is complete if and only if the function $e^{-i\alpha}E(\gamma) - e^{i\alpha}\overline{E(\gamma)}$ does not belong to $\mathbf{Z}_{[-T, T]}$.

One can compute explicitly the functions $A(\gamma)$ and $B(\gamma)$ in some special cases. For instance Dym and Gohberg considered in [10] the case where the spectral density Δ' is the form

$$\Delta'(\gamma) = 1 - \hat{k}(\gamma),$$

where $k \in \mathbf{L}^1(\mathbb{R})$ and such that $1 - \hat{k} > 0$ (in fact, they consider the matrix-valued non Hermitian case). The case where

$$\Delta'(\gamma) = c_H |\gamma|^{1-2H},$$

which corresponds to the fractional Brownian motion, was considered by Dzharidze and H. van Zanten in [12] and will be revisited again in Section 6.2. More generally, one needs to use Kreins's theory of strings, as explained in [11] and [12, Section 2.8], to compute the reproducing kernel of $\mathbf{Z}_{[-T, T]}$.

6. CHAOS AND PREDICTION WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

In this section we now specialize to the case where the spectral measure of X is given by

$$(6.1) \quad d\Delta(\gamma) = C_H |\gamma|^{1-2H} d\gamma, \quad 0 < H < 1,$$

where $C_H = \frac{\Gamma(1+2H) \sin(\pi H)}{2\pi}$ and $\Gamma(x)$ is Euler's Gamma function. This measure satisfies conditions (3.14) and (4.15). The corresponding stationary increment Gaussian process X is called the fractional Brownian motion with Hurst parameter H and is denoted B_H . Its covariance function is given by

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

and it can be shown to have an almost surely continuous sample paths [21]. This process has been found useful in a host of applications, and was extensively studied in the past few decades; see for example [3, 6, 22, 8].

In what follows, we will evaluate the coefficients in the chaos expansion for $B_H(\cdot)$ based on an orthonormal basis with the properties of Theorem 2.1. We start with the case of prediction with respect to the entire past.

6.1. Prediction with respect to the Entire Past ($A = (-\infty, 0]$). For this process $B_H(\cdot)$ with $A = (-\infty, 0]$ we derive the following:

Theorem 6.1. *The outer function in the Wiener-Hopf spectral decomposition of $\frac{\Delta'(\gamma)}{1+\gamma^2}$, where $\Delta'(\gamma) = C_H|\gamma|^{1-2H}$, that admits the reality condition $(-\gamma - i)h(-\gamma) = (\gamma + i)\bar{h}(\gamma)$ is given by*

$$h(\gamma) = \sqrt{C_H} \exp \left\{ i\pi \frac{2H-1}{4} \text{sign}(\gamma) \right\} \frac{i-\gamma}{1+\gamma^2} |\gamma|^{1/2-H}.$$

Proof. Using the formula [9, (11), p. 193], to find the outer factor \tilde{h} in the decomposition (4.18), we see that the outer factor in the factorization of (6.1) is given by

$$\begin{aligned} (6.2) \quad \phi(z) &= \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\gamma z + 1}{\gamma - z} \frac{\log(C_H|\gamma|^{1-2H}/(\gamma^2 + 1))}{\gamma^2 + 1} d\gamma \right\} \\ &= \exp \left\{ -i \left(\pi/2 - \tan^{-1}(z) - z\sqrt{-1/z^2} \tanh^{-1}(1 + \frac{2}{z^2}) \right) \right\} \\ &\quad \times \exp \left\{ -\frac{i}{4\pi} (\log(-z) - \log(z)) ((2H-1)(\log(z) + \log(-z)) - 2\log(C_H)) \right\} \end{aligned}$$

Taking the restriction of $\phi(z)$ to the real line we obtain

$$\frac{i\gamma + 1}{1 + \gamma^2} \sqrt{C_H} \exp \left\{ \text{sign}(\gamma) \frac{2H-1}{4} \pi i \right\} |\gamma|^{1/2-H}.$$

To finish the proof we multiply the last term by i in order to impose the reality condition

$$(-\gamma - i)h(-\gamma) = (\gamma + i)\bar{h}(\gamma) \iff \bar{h}(-\gamma) = -\bar{h}(\gamma).$$

□

The deterministic coefficients in the chaos expansion (4.20) for the conditional expectation $\mathbb{E}[B_H(t)|\mathcal{F}_{(-\infty, 0]}]$ are given by

$$\begin{aligned} r_j^H(t) &\triangleq (\mathbf{1}_t, \xi_j)_\Delta = \int_{-\infty}^{\infty} z_t(\gamma) \frac{\bar{e}_j(\gamma)}{h(\gamma)(\gamma - i)} (1 + \gamma^2) |h(\gamma)|^2 d\gamma \\ &= \frac{1}{\sqrt{\pi C_H}} \int_{-\infty}^{\infty} \frac{e^{i\gamma t} - 1}{\gamma} \left(\frac{1 - i\gamma}{1 + i\gamma} \right)^j \\ &\quad \times \exp \left\{ i\pi \frac{2H-1}{4} \text{sign}(\gamma) \right\} |\gamma|^{H-0.5} d\gamma. \end{aligned}$$

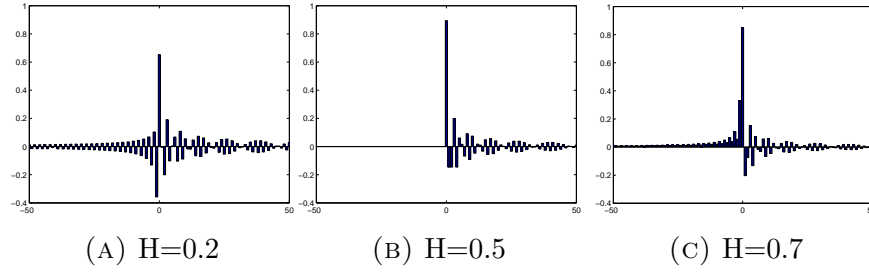


FIGURE 1. The value of the coefficient $r_j^H(1) = (\mathbf{1}_{[0,1]}, \xi_j)_\Delta$, $-100 \leq j \leq 100$, in the sum (6.3) for different values of the Hurst parameter H and $t = 1$. $H = 0.5$ corresponds to the Brownian motion.

See Figure 1 for a graphical illustration of these coefficients for a few cases of the Hurst parameter H .

This gives us

$$(6.3) \quad B_H(t) = \sum_{j \in \mathbb{Z}} r_j^H(t) I(\xi_j),$$

which is a representation of the process $B_H(t)$ as a sum of mutually orthogonal Gaussian random variables independent of time, weighted by the coefficients $\{r_j^H(t), j \in \mathbb{Z}\}$.

It can be shown that the sum (6.3) converges in $\mathbf{L}_2(\Omega)$ uniformly in $t \in \mathbb{R}$. We now split the sum in (6.3) into two sums, of elements of $j < 0$ and $j \geq 0$ respectively, which by (4.20) corresponds to the sum of two (in general not orthogonal) processes:

$$(6.4) \quad \sum_{j \leq -1} r_j^H(t) I(\xi_j) = \mathbb{E} [B_H(t) | \mathcal{F}_{(-\infty, 0]}],$$

and

$$(6.5) \quad \sum_{j \geq 0} r_j^H(t) I(\xi_j) = B_H(t) - \mathbb{E} [B_H(t)].$$

For $t < 0$, $r_j^H = 0$ for any $j \geq 0$, and $B_H(t)$ coincides with (6.4). For $t > 0$, the sum in (6.4) represents what the past ‘thinks’ the future looks like given a specific realization, i.e. the projection of the future on the past. The other part of the sum (6.5) is the complementary projection which can only be determined by the future. Figure 2 illustrates a realization of $B_H(\cdot)$ rendered according to the two sums in (6.4) and (6.5). From (6.5) we also obtain an expression for the prediction error

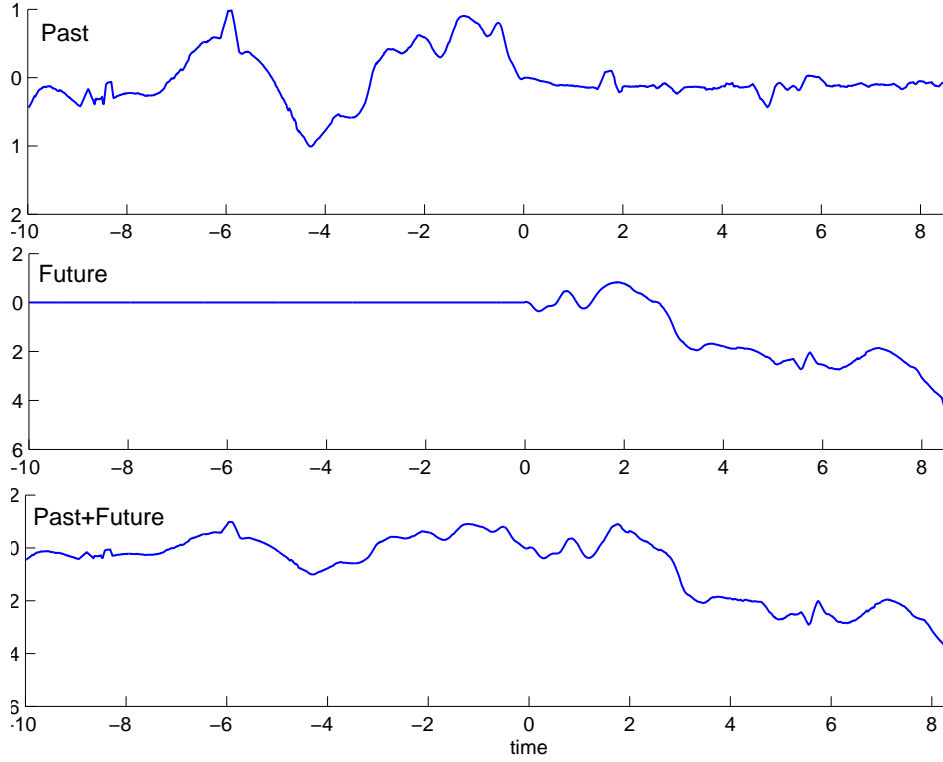


FIGURE 2. One sample path of the fractional Brownian motion B_H with $H = 0.7$ rendered according to the two components of the sum (6.3): The ‘past’ corresponds to the negative indices and the ‘future’ to the non-negative indices.

in estimating $B_H(t)$ for $t > 0$:

$$(6.6) \quad \mathbb{E} \left(B_H(t) - \mathbb{E} [B_H(t) | \mathcal{F}_{(-\infty, 0]}] \right)^2 = \sum_{j \geq 0} (r_j^H(t))^2.$$

A closed form expression for $\mathbb{E} \left(B_H(t) - \mathbb{E} [B_H(t) | \mathcal{F}_{(-\infty, 0]}] \right)^2$ was derived in [14], which leads to the identity

$$(6.7) \quad \sum_{j \geq 0} (r_j^H(t))^2 = \frac{\sin \left(\pi \left(H - \frac{1}{2} \right) \right) \Gamma \left(\frac{3}{2} - H \right)^2}{\pi \left(H - \frac{1}{2} \right) \Gamma(2 - 2H)} t^{2H}.$$

Figure 3 illustrates the prediction error obtained by truncating the sum (6.4) compared to the optimal error (6.7).

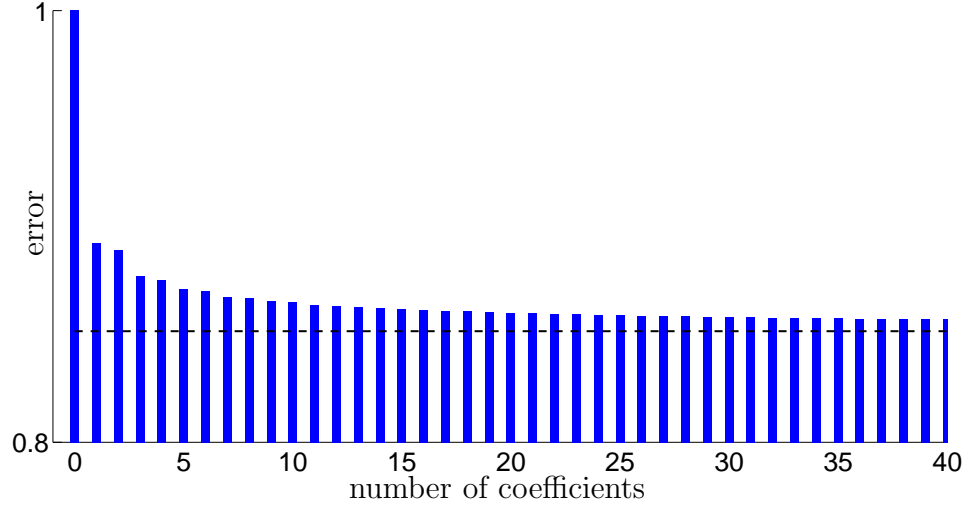


FIGURE 3. Error in estimating $B_H(1)$ with $H = 0.7$ as a function of the number of coefficients in the sum (6.4). The dashed curve is the exact error obtained by (6.7).

The real power of Theorem 2.1 is in simplifying expressions for the condition expectation of a non-linear function of $B_H(t)$. For example, the chaos expansion of $B_H(t)^2$ is found by the Wick product identity [17, Eq. 2.4.10]:

$$\begin{aligned}
 (6.8) \quad B_H^2(t) - |t|^{2H} &=: B_H(t), B_H(t) : \\
 &= \sum_j (r_j^H(t))^2 h_2(E_j) + \sum_{i \neq j} r_i^H(t) r_j^H(t) h_1(E_i) h_1(E_j),
 \end{aligned}$$

where i and j go over all integers, and we used the fact that

$$\sum_{j \in \mathbb{Z}} (r_j^H(t))^2 = \text{var}(B_H(t)) = |t|^{2H}.$$

From Theorem 2.1 we conclude

$$\begin{aligned}
 (6.9) \quad \mathbb{E}[B_H^2(t) | \mathcal{F}_{(-\infty, 0]}] &= \\
 &= |t|^{2H} + \sum_{j \leq -1} (r_j^H(t))^2 h_2(E_j) + \sum_{i \neq j \leq -1} r_i^H(t) r_j^H(t) h_1(E_i) h_1(I(E_j)).
 \end{aligned}$$

6.2. Prediction with respect to $A = [-T, T]$. An orthonormal basis $\{\xi_n, n \in \mathbb{Z}\}$ for the space $\mathbf{Z}_{[-T, T]}$ in the case of the fractional Brownian motion was obtained in [13]. This basis is defined in terms of

the zeros of J_{1-H} , which is the Bessel function of the first order with parameter $1 - H$. Specifically

$$(6.10) \quad \xi_n(\gamma) = \frac{S_T(2\gamma_n/T, \gamma)}{\|S_T(2\gamma_n/T, \cdot)\|_\Delta},$$

where $\dots < \gamma_{-1} < \gamma_0 = 0 < \gamma_1 < \dots$ are the zeros of J_{1-H} . In addition

$$\begin{aligned} S_T(2\eta, 2\gamma) &= S_T(0, 0)(2 - 2H)\Gamma^2(1 - H) \left(\frac{T^2\gamma\eta}{4} \right) e^{iT(\gamma-\eta)} \\ &\quad \times \frac{J_{-H}(T\eta)J_{1-H}(T\gamma) - J_{1-H}(T\eta)J_{-H}(T\gamma)}{T(\gamma - \eta)}, \end{aligned}$$

for $\eta \neq \gamma$, and

$$\begin{aligned} S_T(2\gamma, 2\gamma) &= S_T(0, 0)(2 - 2H)\Gamma^2(1 - H) \left(\frac{T\gamma}{2} \right) \\ &\quad \times \left(J_{1-H}^2(T\gamma) + \frac{2H - 1}{T\gamma} J_{-H}(T\gamma)J_{1-H}(T\gamma) + J_{-H}^2(T\gamma) \right), \end{aligned}$$

for $\gamma \in \mathbb{R}$.

7. CONCLUDING REMARKS

In this work we combined the Wiener chaos decomposition with the problem of linear prediction for Gaussian stationary-increment processes. This was done by considering a special basis for the Gaussian Hilbert space generated by the process, in which each basis element is either completely measurable with respect to the observations or independent of it. This special basis allows us to easily derive the chaos expansion of a random variable conditioned on the σ -field generated by the observations. The result is a chaos approach to prediction, which can be employed to derive an intuitive expression for the chaos expansion of the random variable with respect to the observations. Since the Wiener chaos has been found useful in describing solutions to stochastic differential equations [17, 18], our approach has the potential of better understanding the dynamics of such stochastic systems when past observations of the sample path are given.

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